make. We also outline in Section 4 some of the important issues institutions should consider as they decide which geometry course, or courses, to offer.

1.3 Recommendation 3

The GSG recognizes that geometry plays a special role in the education of future high school mathematics teachers. Teaching teachers is vital, but also vital is our call **not** to reduce geometry to something viewed as necessary only for teacher preparation. In particular, the GSG believes that future high school mathematics teachers are best served by appropriately designed geometry courses for the general mathematics major that also take into consideration the needs of secondary mathematics teachers, not by courses specifically designed only for future teachers. In Section 5 we discuss requirements for the preparation of teachers and give suggestions of how they can be met within courses for the general major.

2 Process

The GSG studied geometry in the undergraduate curriculum in several phases. In addition to internal discussions and a survey of the literature, the group invited geometry instructors to respond to an online survey. We also conducted extensive interviews with a number of respected geometry

Visualization, diagrams, and spatial reasoning. A major mathematical skill is to transfer understanding betw wallpaper groups are all examples of symmetry groups for figures in Euclidean plane geometry, subgroups of the full group of isometries of the Euclidean plane. Symmetry groups provided some of the first instances of the concept of an abstract group in the nineteenth century. Indeed, the transformation groups associated to geometric systems via the *Erlanger Programm*

4 Guiding principles that will inform the design of a geometry course

propositions are still surprising enough that students can appreciate the need for proof. (For more on this subject, see the article by James McClure in the Resources section of this report.)

is tenuous at best. Students must eventually transition from concrete (hands-on) or visual representations to internalized abstract representations. The crucial steps in making such transitions are not clearly understood at present and need to be a focus of learning and curriculum research.

Other researchers and teachers counter the claims of insufficient evidence by citing a vast amount of educational literature (see for instance, [29], [34], [35], [37], [45], [48]). The GSG recommends the use of IGSs and other hands-on activities. While we agree that there is more to be learned about how students transfer knowledge from experiential to abstract, we are persuaded by the existing literature. A huge array of print and online resources can inform teachers who want to incorporate technology or manipulatives into their geometry classes. Examples can be found in the resource section of this report.

Finally, we note that new technologies have historically affected the teaching of geometry, and this is likely to continue in the future. For example, spherical trigonometry, once a staple of a geometry education, was virtually eliminated from the curriculum in the 20th century because navigational computations were now relegated to machines. On the other hand, the rise of video gaming and computer graphics in film makes knowledge of topics like projective geometry and computational geometry a marketable job skill. Twenty years from now, we would like to know: What new skills and understandings might students need to take advantage of new technologies?

4.3 How much emphasis should be placed on transformations?

Felix Klein, in the written version of his 1872 Inaugural Lecture as Professor at Erlangen University, defined geometry as the study of properties of a space that are invariant under a designated group of transformations, the \symmetries" of the geometry. This definition, which provided a unified intellectual structure for all the geometric systems that had appeared by the mid-19th century, had profound effects on the directions of subsequent geometric research. Known as the *Erlanger Programm*, this framework remains today as the definitive definition of a geometric system, both for its clean intellectual elegance and its practical applicability in mathematics and the physical sciences.

This leads us to recommend that any first course in undergraduate geometry have at least some discussion of transformations and the *Erlanger Programm*. Different departments will choose how central a role this should play, and there is no one simple answer; it will vary depending on the geometric topics under consideration, and also on the beliefs and preferences of the faculty. As will be seen in our collection of sample course syllabi in Section 6, the theory of transformations can be developed utilizing either a synthetic or an analytic approach.

Most geometric notions (such as congruence in Euclidean geometry) are elegantly defined by using the designated symmetries of the geometry. Additionally, thinking transformationally | \seeing" the movement of one figure onto another via a symmetry motion | can enhance the intuitive understanding of a problem as well as provide rigorous techniques of proof. Indeed, as will be discussed in Section 5, the Common Core State Standards for Mathematics [CCSS] are increasing the focus on the transformational approach to Euclidean geometry, another reason to mirror this emphasis at the undergraduate level. We believe that any first course in undergraduate geometry should develop enough transformational geometry to support the K{12 curriculum recommendations in the CCSS.

The transformational approach is particularly important when moving beyond Euclidean geometry to study other geometric systems. While other geometries can be de ned and developed axiomatically, it is often di cult to discern the analogies and di erences between the geometries by comparing the corresponding axiomatic systems. However, given the natural ways in which symmetry groups for di ering geometries can be realized via subgroup identi cations, it is then obvious when one geometry is a \subgeometry" of another, making many relationships between the geometries readily apparent.

The study of concrete transformation groups within a geometry course is also excellent preparation for subsequent courses in abstract algebra. Geometry courses stressing a transformational point of view can also point the way to more advanced topics of current importance such as Lie theory and representation theory.

5 Geometry for future high school mathematics teachers

A major portion of the clientele for an undergraduate geometry course has traditionally consisted of future high school mathematics teachers. Their needs should be given particular consideration in designing a geometry course. Our recommendations regarding a geometry course for teachers closely align with those in *The Mathematical Education of Teachers II* [MET II].

The rst and most basic recommendation of MET II is that \Prospective teachers need mathematics courses that develop a solid understanding of the mathematics they will teach."[2] A course in geometry is essential to meeting that goal for high school teachers and the GSG rmly believes that one or more courses in geometry should be a key element of every secondary education mathematics program. An operating principle that applies to all areas of mathematics is that teachers need substantial content at the level one higher than what they will teach, so the college-level geometry course for teachers should go well beyond what is included in the high school syllabus.

There are certain distinct emphases that should be part of a geometry course for future teachers, but that does not necessarily mean there must be a separate *Geometry for Teachers* course. In fact, the GSG believes it would be preferable to have a course that prepares teachers but does transformation that can be written as a composition of basic rigid motions and two figures are *congruent* if there is a congruence that takes one to the other. Once this groundwork is laid, the basic theorems about triangle congruence and similar triangles are proved using transformations.

for Initial Preparation of Mathematics Teachers (NCATE/NCTM, 2003) requires knowledge of the historical development in the number and number systems, algebra, Euclidean and non-Euclidean geometries, calculus, discrete mathematics, probability and statistics, and measurement and measurement system for secondary mathematics teachers, and also includes the recommendation for knowledge of the `contributions from diverse cultures.' Some departments require their majors to take a separate history of math course or a capstone course/project that includes history. Another possibility is to incorporate historical content in the context of geometry courses.

> \When we think about mathematical concepts in our undergraduate courses we often fail to motivate them, to show how concepts emerge from the need to solve problems. This can be shown very easily using historical examples. I'm not suggesting that studying history per se has to be part of the geometry curriculum or even that assessing that has to be part of it, but when we think about what concepts we want to focus on I would opt for ones that show students well their pedigree, where they've come from and why they were needed." Nathalie Sinclair

Real-life applications. Future teachers should understand that geometry research is ongoing and is found in numerous applications. Some faculty incorporate projects so that students research geometry in their daily life. Others highlight the connections of topics to real-life. For instance

> \Mathematics is isolated enough from the real world | we should look for ways to connect it." David Henderson

6 Sample syllabi for a variety of undergraduate geometry courses

There has long been debate about what to teach and how to teach in geometry classrooms, including a tension between practical applications and theoretical considerations that remains today. In some cultures and times the focus of the geometry curriculum was on geometric techniques for practical applications like those in architecture, surveying, and navigation, while in others, it was the axioms of Euclidean geometry that were a fundamental part of a liberal arts education. Geometry teaching continues to evolve with the needs of society as well as to new geometric discoveries in mathematics and mathematics education research.

The first seven courses described below are standard, mainstream geometry courses for which a variety of textbooks exist. We have chosen to include descriptions of two additional courses because they contain interesting ideas that may be of use to those developing new courses. Textbooks do not currently exist for the last two courses described.

We offer our list of syllabi to illustrate the wondrous variety of effective and exciting ways to deliver geometry to our students. The GSG cannot, in good conscience, recommend any one of these courses over the others (though many individual mathematicians will have their own strongly held preferences). We hope our list will stimulate your thinking and will offer useful ideas and guidance to help improve your department's instruction in geometry.

6.1 A survey of geometries

> \Students have to know that geometry is not just one thing. They should understand the distinction between analytic and synthetic approaches, know basics about spherical, hyperbolic, and Euclidean geometries, and, ideally, understand that they are all part of projective geometry." Thomas Bancho

> \The power of geometry comes from being able to think new thoughts and see connections as a result of wide exposure." Tom Sibley

This course aims for breadth, while sacri cing some depth. It assumes that students do remember some of the Euclidean geometry they learned in high school. The level of rigor is purposefully sacri ced in order to develop intuition and to cover some of the breadth of geometry.

In addition to the seven topics listed, the course asks students to carry out an individual discovery project. There is no shortage of possibilities for these: nite geometries, 4-dimensional Euclidean geometry, taxicab geometry, ...

Learning Goals:

Students should demonstrate general understanding of the three major plane geometries: Euclidean, hyperbolic, and spherical.

Students should show expertise in one area through deeper study or a research project.

Students should be able to explain the interplay of synthetic and analytic approaches and to prove theorems at the elementary level using each system.

Students should know enough about transformation geometry to be able to apply their knowledge to classifying patterns by their symmetries.

Students should glimpse the broad picture of how the three major plane geometries are subcases of projective geometry.

Topics:

Euclidean geometry. Building on what students presumably remember from high school, the course should develop strength in three areas of Euclidean geometry: The ability to prove basic theorems from axioms; knowledge of representative constructions, such as the various triangle center constructions; and a broad understanding of the historical narrative of Euclidean plane geometry, from Euclid to Hilbert. The role of Euclids Fifth Postulate (and its equivalents) is highlighted.

Analytic geometry. Students should understand the analytic method as a powerful alternative to synthetic reasoning. For instance, students should be able to use the concept of slope to prove that the midpoints of a quadrilateral form a parallelogram.

Hyperbolic geometry. Using an axiomatic approach, possibly aided by one of the standard models, students should learn how the negation of Euclids Fifth Postulate leads to a rich body of theorems. These should include the non-existence of similar triangles and the fact that the angles in a triangle sum to less than 180 degrees.

Spherical geometry. With only as much detail as time permits, students should understand the sphere (and its quotient by the antipodal map), both as a physically useful space and a model to illustrate a third alternative system of geometry. (If time permits, the sphere provides an interesting ground for debating the fine points of axiom systems.)

Transformations. Whether through connection to linear algebra, complex numbers, or a synthetic approach, students should learn the vocabulary of Euclidean translations, rotations, reflections, and glides. The large goal is to understand the structure of isometries as being direct or indirect, and as always being the product of at most three reflections.

Symmetries. The knowledge of transformations is applied to an overview of symmetry groups in the plane, including rosettes, friezes, and wallpaper patterns.

(following Hilbert). It is much easier to make connections with high school geometry when the real numbers and measurement are allowed, so axioms based on the real numbers are usually preferable for a course populated with future high school teachers. Hilbert's axioms are probably preferable for students who will go on to graduate work in mathematics.

Neutral geometry. This is the part of 2-dimensional geometry in which no parallel postulate is assumed. Many standard results, such as the triangle congruence conditions, can be proved in this setting. Students should be able to give axiomatic proofs that various familiar statements are equivalent to Euclid's Fifth Postulate and they should have a clear understanding of what this \equivalence" means. (It does not mean that the statements are logically equivalent to the parallel postulate in isolation, but only that they are equivalent in a context in which other assumptions have been made.)

Euclidean geometry. Students should learn how the basic results of Euclidean geometry (such as theorems about similar triangles and the Pythagorean Theorem) relate to the parallel postulate.

Hyperbolic geometry. Students should learn to give axiomatic proofs of results in hyperbolic geometry and understand that there are two essentially di erent kinds of parallel lines in the hyperbolic plane.

Models. In order to demonstrate the independence of the parallel postulate, students should be familiar with models for Euclidean, hyperbolic, and spherical geometries and they should have experience working in all three of those geometries.

Transformations. The ways in which transformations interact with the formulation of the axioms can be explored, especially in a course for future high school teachers.

Other axiom systems. As time allows, instructors may include other axiom systems, selecting from among those for nite geometry, projective geometry, spherical geometry, or origami.

6.3 Euclidean geometry

Euclidean geometry has continued to expand and develop since the time of Euclid. The subject contains many of the most surprising and beautiful results of elementary mathematics. Euler initiated a revival of the subject in the eighteenth century when he discovered that the three classical triangle centers are collinear. Since that time many mathematicians have contributed to its development. Euclidean geometry is currently experiencing another revival and is the subject of intense research even today, probably because dynamic geometry software makes it simple to construct and explore intricate diagrams.

Learning Goals:

Students should learn that Euclidean geometry continued to develop after the time of Euclid and that the subject is still expanding today.

Students should learn the basic results and techniques of post-Euclid Euclidean geometry. They should come to appreciate the great beauty of the results as well as how surprising some of them are.

Students should learn to prove theorems in Euclidean geometry. They can assume the basic results of high school geometry and build from there. The Euclidean geometry course is a natural setting in which to learn to appreciate the power of logical deduction since the results in the subject are non-obvious but still susceptible to proofs that are relatively simple in their logical structure and easy to understand.

Students should become familiar with dynamic geometry software. They should learn to use the software to discover, explore, and illustrate the results of Euclidean geometry. Students should be able to make tools of their own that perform standard constructions. (For example, they should be able to make a tool that constructs the circumscribed circle of a triangle.)

Topics:

A review of elementary Euclidean geometry. Given the great differences in high school geometry courses, it is necessary to review the basic facts and techniques of high school Euclidean geometry.

Triangle centers. The three classical triangle centers (centroid, orthocenter, and circumcenter) as well as the Euler line should definitely be included. In addition, a sampling of other triangle centers can be discussed — see the *Encyclopedia of Triangle Centers* listed in the Resources section of this report.

Circumscribed, inscribed, and exscribed (or escribed) circles. The course should cover both the construction of these circles and proofs that they exist.

The theorems of Ceva and Menalaus. A good place to discuss duality and to make connections with projective geometry.

Transformations. A proof of the classification of rigid motions of the plane could be included.

Constructions. See the discussion of constructions in §3 of this report.

Other topics. There are many additional topics that could be included: The nine point circle and Feuerbach's theorem, the theorems Miquel, Morley, Desargue, Brianchon, Pappus, Simson, and Ptolemy, as well as Pascal's Mystic Hexagram.

6.4 Transformational Euclidean geometry

A transformational approach to establishing geometric results, even when restricted solely to Euclidean geometry, is a powerful tool for developing proofs, solving geometric problems, and developing deep geometric understanding. While transformational geometry is closely associated with the use of analytic (most commonly linear algebraic) techniques, it is indeed possible to develop transformational geometry using a synthetic approach. A synthetic development is particularly effective when studying Euclidean geometry.

Learning goals:

Students should learn how to approach geometric problems from a transformational viewpoint and to visualize the movement inherent in a transformational approach.

Students should learn the basic history of the development of geometry, at least enough to appreciate the reasons leading to Felix Klein's formulation of the *Erlanger Programm* in 1872. They should understand the uni cation achieved by this point-of-view.

Students should understand and master the use of the group structure of the collection of symmetries of a geometry.

The transformational approach allows a clean and elegant method for comparing different geometries via containment relationships between the different groups of symmetry transformations. For example, similarity geometry is a subgeometry of affine geometry, which in turn is a subgeometry of projective geometry. This allows for a logical approach to the classification of invariants in each geometric system, as well as for a clearer understanding of how to assign particular theorems to their "proper" geometries.

When focusing on geometries beyond Euclidean we feel that an analytic approach via linear algebra is probably more effective than a synthetic approach.

Learning goals:

- Students should learn how to approach geometric problems from a transformational viewpoint and to visualize the movement inherent in a transformational approach.

- Students should learn the basic history of the development of geometry, at least enough to appreciate the reasons leading to Felix Klein's formulation of the *Erlanger Programm* in 1872. They should understand the unification achieved by this point-of-view.

- Students should understand and master the use of the group structure of the collection of symmetries of a geometry.

- Students should develop facility with the basics of the real projective plane and its symmetries.

- Students must develop an intuitive understanding of projective duality and its embodiment in the linear algebra model of the real projective plane.

- Students should come to appreciate Arthur Cayley's (slightly exaggerated) claim that *"projective geometry is all geometry"* by seeing the other classical plane geometries as subgeometries of the real projective plane.

Topics:

- The Real Projective Plane RP^2. Emphasis is recommended on the analytic model as equivalence classes of non-zero vectors in R^3. Development of the linear algebraic tools for handling incidence relations and the cross-ratio. The group of projective transformations, developed as collineations. Perspectivities. Four-fold transitivity of projective transformations. Theorems of Desargues and Pappus. Projective conics.

- The Affine Plane. Motivated and modeled via projectivization, that is, embedding as the plane $z = 1$ in R^3 when R^3 is converted to RP^2. The group of affine transformations seen as projective transformations fixing the line at infinity. The invariance of the affine ratio. The centroid of a triangle. Barycentric coordinates. Three-fold transitivity of affine transformations. Theorems of Ceva and Menelaus. Classification of affine conics.

- The Euclidean Plane: Isometries and Similarities. Developed as subgeometries of affine geometry and projective geometry. The invariance of angle measure and distance ratios in similarity geometry. The invariance of distance in congruence geometry. Examples of Euclidean theorems that are not affine theorems. Classification of conics under similarities and isometries. Laguerre's Formula for angle measure between lines.

Hyperbolic and Elliptic Geometries. Developed as subgeometries of the real projective plane via the choice of the \absolute" projective conic. Distance and angle measure obtained from the cross-ratio and the absolute conic via the Cayley-Klein method. The subgroups of hyperbolic and elliptic symmetries on the hyperboloid and spherical models. The Beltrami-Klein disk model for hyperbolic geometry. A selection of theorems in hyperbolic and elliptic geometry.

If time permits: The Complex Projective Line CP^1 and Inversive Geometry. Inversions, circular transformations, and Möbius transformations. The Riemann sphere and stereographic projection. The cross-ratio in CP^1. Feuerbach's Theorem. Hyperbolic and Elliptic geometry as subgeometries of CP^1. The Poincare disk and upper half-plane models for hyperbolic geometry.

If time permits: The Geometry of Space-time. The Poincare and Lorentz groups as the symmetry groups for 4-dimensional space-time in special relativity.

6.6 Computational geometry

Although geometry is as old as mathematics itself, discrete geometry only fully emerged in the 20th century, and computational geometry was only christened in the late 1970s. The terms \discrete" and \computational" t well together as the geometry must be discretized in preparation for computations. \Discrete" here means concentration on nite sets of points, lines, triangles, and other geometric objects, and is used to contrast with \continuous" geometry, for example, smooth surfaces. Although the two endeavors were growing naturally on their own, it has been the interaction between discrete and computational geometry that has generated the most excitement, with each advance in one eld spurring an advance in the other. The interaction also draws upon two traditions: theoretical pursuits in pure mathematics and applications-driven directions, often arising in computer science. The con uence has made the topic an ideal bridge between mathematics and computer science.

Learning Goals: The eld has expanded greatly since its origins and now the new connections to areas of mathematics (such as algebraic topology) and new application areas (such as data mining) seems only to be accelerating.

Students must show understanding of the core pillars of this subject: polygons, convex hulls, triangulations, and Voronoi diagrams.

Students must grasp questions from algorithmic standpoints, not just addressing whether something can be done, but how it can be constructed, and how e cient such constructions can be.

Students should become comfortable experimenting with (freely available) applets and programs that emphasize the applicability of the subject.

Interplay between algorithms should not be ignored. There are numerous areas related to this topic, as it bridges mathematics and computer science.

To experience the richness of this growing eld, students should pursue in depth one or two extra topics not classically covered among the topics below.

Topics:

Polygons: We introduce the worlds of the \discrete" and the \computational" to a mathematical audience. The key tool will be the study of polygons and polyhedra, the building blocks of 2D and 3D discrete geometry. Topics will include triangulations, enumerations, dissections, and art gallery theorems.

Convex Hulls: Although a convex hull of a set of points in the plane is easy enough to de ne, how does one go about computing it? What does it mean to construct a geometric algorithm, and how can one measure better algorithms? We look at several powerful algorithms for 2D hulls, and glimpse into the di culties with 3D hulls, along with framing the big-Oh notation. In particular, consider Incremental, Divide-and-Conquer, Gift Wrapping, and Grialgorithmsoem(rappiSne)-

- Geometry of curves in space, including the Frenet frame
- Theory of surfaces, including parameterizations, first and second fundamental forms, curvature and geodesics
- The concluding part of the course could be a focus that depends on the interest of the instructor and students, such as the Gauss-Bonnet Theorem, the theory of minimal surfaces, or the geometry of space-time with applications to general relativity.

Ideally a course in differential geometry allows students to see the connections between such topics as calculus, geometry, spatial visualization, linear algebra, differential equations, and complex variables, as well as various topics from the sciences, including physics. The course may serve as an introduction to these topics or a review of them. The course is not only for mathematics majors — it encompasses techniques and ideas relevant to many students in the sciences, such as physics and computer science.

One inescapable prerequisite for this course is multivariable calculus. Some schools have successfully taught differential geometry with nothing more than multivariable calculus as a prerequisite, and so this is a feasible single requirement, especially if computer algebra software will be utilized for computations and visualization. The immediate benefit is that more students in other majors could take the course. However, students coming out of a multivariable calculus course may not have the mathematical maturity needed for the course, depending on the focus and level. Some schools require linear algebra, a proof-writing course, and/or differential equations as additional prerequisites for the class.

6.8 Geometric structures: Axiomatics, graphs, polygons, polyhedra, and surfaces

Joseph Malkevitch teaches a course at York College (CUNY) whose goal is to cover as broad a range of topics with geometrical flavor as possible in one course. The desire is to have students explore geometry and hence become more broadly aware of geometric phenomena and applications of geometry. This course clearly and purposely emphasizes breath over depth. Malkevitch believes that it's much easier for a student to tackle a mathematical topic if he or she has been exposed earlier to some of the basic concepts and results.

Learning Goals:

- Students will come to understand the rudiments of axiomatic systems via studying finite affine, finite projective, and finite hyperbolic planes, as well as by comparing Desarguesian and non-Desarguesian planes.
- Students should be able to explain how the real projective plane RP^2 is constructed from the Euclidean plane and reasons for the central importance of homogeneous coordinates for RP^2.
- Students will gain exposure to planar graphs and Euler's polyhedral formula, which further requires the Jordan Curve Theorem. This leads into a proof of the existence of the five platonic solids and a discussion of frieze and wallpaper patterns.
- Students will gain facility with basic ideas concerning surfaces via explorations of Moebius strips, spheres with handles, and nets of polyhedra.

Students will consider the concept of distance via extensive work with Taxicab geometry.

Topics:

What is Geometry? Geometry as the study of space, shapes, and visual phenomena. the role of careful looking. Geometry as a branch of mathematics and as a branch of physics.

De nitions, Axioms, and Models. Di erent kinds of geometry, Euclidean geometry, Bolyai-Lobachevsky geometry, projective geometry, a ne geometry, and taxicab geometry. Axioms systems and rule systems in sports.

Graph Theory. The uni er of course topics.

Geometrical transformations. Translations, rotations, re ections, shears, homothetic mappings, projective transformations, applications to computer vision and robotics, Felix Klein's

Learning Goals

Students will explore several networks of de nitions and propositions on the plane, the sphere and other surfaces, using multiple approaches drawing on transformations symmetry and isometries, supported by dynamic geometry programs, physical manipulatives and objects, and methods of visual and spatial reasoning;

Students will gain facility with basic ideas of groups of isometries generated by products of re ections, and the application of these ideas in geometry, including Klein's Hierarchy of geometries, in algebra and in various applied settings;

Students will re ect on presentations, experiences and course readings connected to (i) the signi cance and development of mathematical reasoning through multiple representations and switching strategies; (ii) spatial reasoning and visual reasoning, as well as illustrations of embodied cognition (iii) \folding back" as a learning strategy (Pirie-Kieran), and (iv) the key

Develop the ability to represent all plane isometries with 3x3 matrices using barycentric coordinates, and all spherical isometries with 3x3 orthonormal matrices, and connect key features of the isometry with the eigenvectors of these matrices, both numerically and visually, and solve additional geometric problems with these reasoning tools. Connect these matrix representations to trigonometry identities as well as current methods in computer graphics;

Distinguish chiral and achiral shapes, including molecules, and their impact on patterns in 2 and 3 dimensions, including biochemistry such as Vitamin E, drug design;

Explore the properties of the sum of the internal and external angles of polygons, by motions, scaling (in the plane), dynamic geometry programs, and by holonomy and areas of polygons on the sphere (discrete Gauss-Bonnet), using dynamic geometry programs and proofs with parallel transport;

Explore `Parallel Postulates' in the presence of the first four Euclidean postulates, through the equivalence of various `parallel properties' in the plane (and how each breaks or is modified in spherical geometry);

Klein's Hierarchy of Geometries, including the fundamental role of groups of transformations, and corresponding invariants;

Spherical Polyhehdra, supported by proof(s) of Euler's formula and extensions to Descartes' formula for polyhedral angle deficits (another form of discrete Gauss-Bonnet), with illustrations through Platonic solids (built with Polydron) and when the formula fails or needs modification. Possible extension to 4-D (through projects).

7 Resources

Remark: *The presence of a text on this list is not meant to imply an endorsement of that text, nor is the absence of a particular text from the list meant to be an anti-endorsement. The texts are chosen to illustrate the sorts of texts that support the various types of geometry courses described in this report. Please note that some of the books listed below were written by the authors of this report.*

Suggested textbooks for *A survey of geometries*

1. Henderson, David W. and Daina Taimina, *Experiencing Geometry, Euclidean and Non-Euclidean with History*, 3rd edition, Pearson, Upper Saddle River, NJ, 2005.

2. Sibley, Thomas Q. *Thinking Geometrically: A survey of Geometries*. Mathematical Association of America, Washington DC, 2015

Suggested textbooks for *Axiomatic Geometry*:

3. Greenberg, Marvin Jay, *Euclidean and Non-Euclidean Geometries: Development and History* (4th edition), Freeman, 2008 (based on Hilbert's axioms).

4. Hartshorne, Robin, *Geometry: Euclid and Beyond*, Springer, 2000 (based on Hilbert's axioms).

5. Lee, John M., *Axiomatic Geometry*, American Mathematical Society, 2013 (based on metric axioms).

6. Venema, Gerard A., *Foundations of Geometry* (2nd edition), Pearson, 2001 (based on metric axioms).

Suggested textbooks for *Euclidean geometry:*

7. Coxeter, H. S. M. and S. L. Grietzer , *Geometry Revisited*. MAA, 1967. (A classic, but not written in the style of modern textbooks.)

8. Isaacs, Martin, *Geometry for College Students*. Brooks Cole 2000.

Suggested textbooks for *Transformational Euclidean geometry*:

9. Barker, William and Roger Howe, *Continuous Symmetry*. Providence, RI: American Mathematical Society, 2007.

10. Martin, George E., *Transformation Geometry: An Introduction to Symmetry*, New York: Springer-Verlag, 1982.

Suggested textbooks for *Transformational geometry: beyond Euclidean*:

11. Brannan, David A., Matthew F. Esplen, and Jeremy J. Gray, *Geometry*

Suggested resources for *Exploring geometries with hand and eye:* There currently are no textbooks that cover all the topics suggested for this course. The course draws materials from the following texts:

24. Henderson, David W. and Daina Taimina, *Experiencing Geometry, Euclidean and Non-Euclidean with History*, 3rd edition, Pearson, Upper Saddle River, NJ, 2005.
25. Tall, David et al, *Cognitive Development of Proof* in New ICMI Study Series, Vol. 15 Proof and Proving in Mathematics Education, Michael de Villiers and Gila Hanna (eds) 2012.
26. Whiteley, Walter , *Learning to see Like a Mathematician. In Multidisciplinary Approaches to Visual Representation and Interpretation* (G. Malcom Ed), Elsevier 2005, 279-292.

Other materials are available directly from Walter Whiteley (whiteley@mathstat.yorku.ca) and