Ordinary Di erential Equations.

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Introduction. This is a report from the working group charged with making recommendations for the undergraduate curriculum in di erential equations. As discussed below, the basic sophomore level Di erential Equations course has changed dramatically (at least at some institutions) over the past twenty years, mainly due to the much wider availiblility of computer resources. We discuss a number of these changes below and then provide three di erent syllabi, each aimed at a slightly di erent audience. Finally, we provide some references including sample texts.

How Di erential Equations Courses Have Evolved. As part of our committee work, we surveyed a half dozen di erent schools regarding their sophomore-level Ordinary Di erential Equations (ODE) course. These schools included two-year colleges, liberal arts colleges, and universities. Two things became clear from this survey. The rst is that the ODE course has undergone a remarkable transformation over the past twenty years. The second is that there is now no set curriculum for this course.

Regarding the rst point, the ODE course previously focused almost exclusively on speci c analytic methods for solving di erential equations. Often, examples of di erential equations were simpli ed so that the corresponding equation could then be solved explicitly (as, for example, the nonlinear pendulum equation was often linearized to accomplish this). Usually, the types of equations covered in the course were rst and second (and maybe higher) order di erential equations but rarely included systems of di erential equations.

The major change in the ODE courses over the past two decades has been primarily motivated by the availability of the computer. Currently, most courses use some of the many di erent software packages that are availa on the project can also be required. Fortunately, almost all ODE textbooks nowadays contain topics for such projects, and there is also plenty of information on modeling applications and lab projects on-line. For example, see the CODEE website.

New Topics to Include. Another advantage of the computational approach is that this allows for the inclusion of topics that have not usually been treated in an introductory ODE course. For example, since visualization of solutions is much more important these days, it is natural to convert second order linear differential equations to planar systems of differential equations, so solutions can then be visualized in the phase plane. Consequently, linear systems of the form $Y' = AY$ are now included in most courses. Usually, only planar systems are covered, so A is a 2 by 2 matrix, although more advanced courses often introduce higher dimensional systems as well. Therefore the new course includes a relatively brief introduction to linear algebraic topics such as matrices, eigenvalues, and eigenvectors. Often these topics are introduced gradually during the course, not necessarily as a week or two week long diversion into linear algebra. The inclusion of these topics connects well with the linear algebra course that many students often take after the ODE course, as it provides a nice visual application of linear algebra. For example, the most important solutions of linear systems are the "straight line" solutions. These are solutions of the form $e^{\lambda t}V$, where λ is an eigenvalue of A and V is its associated eigenvector, so students see early on why eigenvalues and eigenvectors areimportant.

In addition, nonlinear systems are often covered and this necessitates using a qualitative approach to understanding the system as most nonlinear systems cannot be solved analytically. Topics such as nullclines and linearization near equilibria provide some of the tools necessary for this geometric approach. Hamiltonian and gradient systems may also be introduced, and this provides further tools. Some courses even occasionally include modern topics such as chaos and bifurcation theory.

Computational Tools. There is no set standard for how to use computers in the ODE courses. Many institutions use computer algebra systems such as Maple or Mathematica. This is fine when students already have some familiarity with these tools, but often certain students in the course do not come with this background and hence they encounter a steeper learning curve. Another resource is that many of the newer ODE texts come with software specifically designed for topics included in the book. The advantage here is that students do not have to take time to learn how to use the software. A third possibility is the use of spreadsheets. When students try to learn the numerical algorithms for approximating solutions to ODEs (like Euler's Method or Runge-Kutta 4), a natural method to encode these algorithms involves a spreadsheet. Moreover, the graphical capabilities of spreadsheets also help students visualize the outputs of these algorithms. A major advantage of this approach is that almost all students in the ODE course already have a good background using spreadsheets.

What to Eliminate. Given the inclusion mentioned above of more modeling/research projects, topics from linear algebra, and more computational based topics in the contemporary ODE courses, clearly some topics from the traditional ODE course must be dropped. Most modern courses now eliminate some or many of the specialized analytic methods for solving ODEs. For example, integrating factors, variation of parameters, solutions of special equations like the Bernoulli equation, and other such techniques are sometimes not included. Instead, in some instances, the only types of first order ODEs that are now solved analytically are linear and separable equations. Also, series solutions are sometimes eliminated (though, of course, the linearization techniques mentioned above do involve the "first" terms in such a series solution). And, except in courses heavily populated by engineering students, Laplace transforms can also be eliminated.

Cognitive Learning Goals

A. **The Culmination of Calculus.** ODEs really form the primary basis for the study of calculus and so this course should strive to bring together many of the previously covered concepts in a way that confirms their usefulness and necessity.

B. **Use of Technology.** The ODE course is easily the course in the introductory undergraduate mathematics curriculum in which the use of technology is most essential. Students should be encouraged to use these tools in homework, in projects, and in simply visualizing the various qualitative aspects of ODEs.

C. **Applications.** There are major applications involving di erential equations in all areas of science and engineering, and so many of these should be included in the ODE course to show students the relevance and importance of this topic. Some applications include mass-spring systems, forced, damped, and undamped pendulum equations, and Newton's Laws (physics), electrical circuits (engineering), enzymatic reactions (chemistry), population models (biology), Kepler's Laws (astronomy), compound interest models (economics), and the Lorenz system (meteorology).

D. **Introduction to Higher Level Mathematics.** This course also provides an opportunity for students to get a glimpse of some topics in higher level mathematics courses. Examples include linear algebra (solving linear systems of ODEs and linearization), numerical analysis (understanding numerical methods such as Euler's method or RK4 for approximating solutions of ODEs), real analysis (the existence and uniqueness theorem), and dynamical systems (bifurcation theory and chaos).

Sample Syllabi. Because of the variety of aforementioned topics and styles, there is now no one set curriculum for an ODE course. Each of the schools we surveyed had di erent approaches to the course. Below is a summary of the syllabi of the three types of ODE courses o ered at the collegiate level: the captsone college course, the sophomore level service course, and the course aimed primarily at math majors.

1. The capstone ODE course not necessarily aimed at partner disciplines. Many, if not all, community colleges o er Di erential Equations as the capstone course in the mathematics department. Some time is devoted to reviewing relevant topics in single variable calculus. The course would not qualify as a replacement for a junior-level introductory ODE course o ered at a four-year college or a university. It might, however, qualify as a replacement for a service ODE course o ered

in the freshman or sophomore years at those institutions. Here is an outline of a syllabus for such a 13-week semester ODE course.

> First order ODEs: linear equations, radioactive decay model, separable equations, review of relevant topics from single-variable calculus, slope fields, classifying equilibrium points of autonomous first order ODEs, existence and uniqueness theorems, introduction to the use of solvers, Euler's Method, population models. (4 weeks)
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> Second order ODEs: solution techniques for linear ODEs with constant coefficients, review of complex numbers, direction fields, mass-spring model, forced and damped harmonic motion, beats and resonance. (4 weeks)
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> Planar systems: solution techniques for planar linear systems with constant coefficients, an introduction to relevant topics from linear algebra (matrices, eigenvalues, eigenvectors), nonlinear autonomous systems, classifying equilibrium points, phase portraits, various models. (5 weeks)

2. An ODE course which is part of the math core required of science and engineering students taken in the freshman and sophomore years. That is, this course is really a service course. Unlike the previous syllabus, this course will include many more applications in the client disciplines. As in the earlier course, students will have not taken a course on Linear Algebra prior to this course, so the syllabus allots time to cover the linear algebra topics needed for solving linear systems. Laplace transforms are included. The modeling projects emphasize the usability of ODEs in other disciplines such as biology (predator-prey and competing species models), electrical engineering (circuit theory models), and physics (mechanical systems, the n-body problem). The syllabus for such a 13-week semester course might look something like this:

> First-order ODEs: Slope fields, separable equations, linear equations, nonlinear ODEs, existence and uniqueness, numerical solutions, qualitative analysis of solutions. (3 weeks)

Second-order ODEs: Linear ODEs with constant coefficients, forced and damped harmonic oscillator, beats and resonance, phase plane, applications to springs, electrical circuits. (3 weeks)

Laplace Transforms: Definition, solving initial value problems for linear constant-coefficient second-order ODEs. (2 weeks)

3. Borelli, R.L., Coleman, C.S., Di erential Equations: A Modeling Perspective, John Wiley and Sons, Inc.

4. Boyce, W.E., DiPrima, R.C., Elementary Di erential Equations, John Wiley and Sons, Inc.

5. Braun, M. Di erential Equationsnc.

15. [SIMIODE] - Systemic Initiative for Modeling Investigations and Opportunities with Differential Equations

Articles from Mathematics Education Research:

16. Rasmussen, Chris and Oh Nam Kwon, An inquiry-oriented approach to undergraduate mathematics, *Journal of Mathematical Behavior* 26 (2007), 189-194.

    This article appeared in a special issue of the *Journal of Mathematical Behavior*. In addition to providing an overview of the five articles in the issue, the authors highlight the theoretical background for an innovative approach in differential equations called the Inquiry Oriented Differential Equations (IO-DE) project. and provide a summary of two quantitative studies done to assess the effectiveness of the IO-DE project on student learning.