Transitions to Proof

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General Information

The ***Title of the Course*** varies widely. We encountered:

rigorous arguments; communicate mathematical ideas clearly and coherently both verbally and in writing; approach mathematical problems with curiosity and creativity and persist in the face of difficulties; work creatively and self-sufficiently with mathematics."

Such skills are crucial in mathematics and so these goals seem unobjectionable. Unfortunately, jam-packed syllabi can create a tension between the imperative to cover content and giving students *time* to wrap their minds around the mathematics in these important ways. All too often important cognitive goals give way to making sure our students "see" important mathematical ideas. As instructors, we may close our eyes and cross our fingers, hoping that our students are coming to grips with the details outside of class time. A few students do, picking up analytical and critical thinking skills by osmosis. Most students can't, however, because they have no idea how to go about it, or (worse) don't know what it means to do so. Such students can sometimes get through lower-level courses by imitation, but struggle in upper-level courses that require them to think abstractly, construct logical arguments, and use mathematical language precisely. It was this observation that led to the proliferation of so-called "transition" courses, which were rare in the early 1990's but now are quite common. The primary purpose of a transition course is to ramp up students' abilities to think and approach probl ip e mathematicians, providing a cognitive bridge between more procedural lower-level courses such as Calculus and upper-level abstract courses such as Real Analysis, Probability Theory, or Abstract Algebra. In transition courses, content goals take a back seat; the primary goals of the course are cognitive. Where time constraints cause tension between cognitive goals and content coverage goals, content should *always* give way to activities that help students progress in developing analytical, critical-reasoning, problem-solving, and communication skills and acquiring mathematical habits of mind.

Transition courses are, of course, not devoid of mathematical content. If students are to reason carefully, think critically, solve problems, and communicate mathematical ideas precisely, they must have ideas to grapple with, problems to solve, and opportunities to tal M p nd write about mathematics. However, the choice of mathematical "context" varies quite a bit. Many institutions teach a course centered on standard "mathematical building blocks" such as sets, relations, functions, and so forth; others introduce students to mathematical reasoning in the context of specific subject matter. Some elementary Discrete M0450056239.934(maticd c)5d sesfoour inst ceintcestudentl t

Central Goals

The course should concentrate on training students in clear thinking and creative experimentation in the exploration of mathematical ideas. Because proof solidifies intuition into certainty, the course should also focus on the careful use mathematical language, logical reasoning and proof. The course should concentrate on imparting to students:

- the ability to read, understand, and construct proofs;
- the ability to write and speak about mathematics using precise mathematical language;
- an understanding the role of definitions in mathematics and being able to use (and possibly construct) them effectively;
- a basic understanding of elementary logical principles and proof techniques. (Examples include the proper use of logical connectives and quantifiers, negation of mathematical statements, the equivalence of a statement and its contrapositive, direct proof,

This study gives important insight into good pedagogy for all mathematics courses. Unfortunately, student-centered activities take a great deal of time and may consequently be crowded out by the “coverage” imperatives of content-driven courses. (As students learn mathematics by doing mathematics, this is unfortunate, but it certainly happens.) However, as we have already noted, the most important goals of the transition course are cognitive goals. Therefore a well-constructed syllabus for a transition course should always be “lean” enough in terms of content that students are actively engaged in the material at every step of the way---both in class and outside of class.

Sample content lists

It is not really important what mathematical “context” is used to teach mathematical reasoning and proof. We emphasize again that the main imperative for the course is to give students many *opportunities*

2. Burger, Edward B., Extending the Frontiers of Mathematics: Inquiries into Proof and Argumentation, Wiley, 2008.

Comments: This book is structured as a long series of interconnected problems, made up of statements that may or may not be true---the instructions to the student are frequently to “prove and extend” or “disprove and salvage.” Thus it supports an inquiry-based approach, and particularly encourages students to probe and conjecture. The book includes chapters on selected topics in Number Theory, Discrete mathematics, Algebra, and Analysis.

3. Schumacher, Carol, *Chapter Zero: fundamental notions of abstract mathematics*, 2nd Edition, Addison-Wesley, 2001.

Comments: This book supports an inquiry-based approach. Thus it contains very few finished proofs, so it is structured as a long series of problems that are left for the students. On the other hand, it supports the students’ mathematical development by helping them explore the motivation that underlies the ideas and by giving them practical tips about proof techniques and the construction of arguments. Its discussion of logical principles and proof techniques is brief and informal. Also includes chapters on selected elementary topics in Number Theory, and the Real Number System.

4. Smith, Douglas., Maurice Eggen, and Richard St. Andre, *A Transi*[illegible],

9. Edwards, Barbara S. and Michael B. Ward, Surprises from Mathematics Education Research: Student (Mis)use of Mathematical Definitions, , 111(5), 411-424.

: When Edwards reported that she had found that the definitions of “limit” and “continuity” were problematic for some of the real analysis students, Ward’s intuitive reaction was that those words were “loaded” with connotations from their nonmathematical use and from their less than completely rigorous use in elementary calculus. He said, “I’ll bet students have less difficulty or, at least, different difficulties with definitions in abstract algebra. The words there, like ‘group’ and ‘coset,’ are not so loaded.” So the authors decided to study student understanding and use of definitions in Ward’s own introductory abstract algebra. Ward was surprised to see his algebra students having difficulties very similar to those of Edwards’s analysis students. In particular, he was surprised to see difficulties arising from the students’ understanding of the very nature of mathematical definitions, not just from the content of the definitions. This article reports

12. Iannone, P., Inglis, M., Mejıa-Ramos, J. P., Simpson, A. & Weber K. (2010). Does generating examples aid proof production?
, 1-14.

: Many mathematics education researchers have suggested that asking learners to generate examples of mathematical concepts is an effective way of learning about novel concepts. To date, however, this suggestion has limited empirical support. Undergraduate students were asked to study a novel concept by either tackling example generation tasks, or reading worked solutions to these tasks. However, there was no advantage for the example generation group on subsequent proof production tasks. The

15. Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? , (1), 4-36.

: The authors report on an exploratory study of the way eight mid-level undergraduate mathematics majors read and reflected on four student-generated arguments purported to be proofs of a single theorem. The results suggest that mid-level undergraduates tend to focus on surface features of such arguments and that their ability to determine whether arguments are proofs is very limited -- perhaps more so than either they or their instructors recognize. They begin by discussing arguments (purported proofs) and of those arguments, that is, reflections of individuals checking whether such arguments really are proofs of theorems. The authors provide a detailed analysis of the four student-generated arguments and finally analyze the eight students' validations of them.

16. Selden, A. & Selden, J. (2008). Overcoming students' difficulties in learning to understand and construct proofs. In M. P. Carlson and C. Rasmussen (Eds.), (pp. 95-110), MAA Notes Volume No. 73. Washington, DC: Mathematical Association of America.

: This chapter provides an overview of students' difficulties in learning to understand and construct proof. The major sections are titled: the curriculum and students' and teachers' conceptions of proof, understanding and using definitions and theorems, knowing how to read and check proofs, knowing and using relevant concepts, bringing appropriate knowledge to mind, knowing what's important and useful, and teaching proof and proving.

17. Tall, D. (1998). The cognitive development of proof: Is mathematical proof for all or for some? In Z. Usiskin (Ed.), (Vol. 4, pp. 117–136). Reston, Virginia: NCTM. Also: http://homepages.warwick.ac.uk/staff/David.Tall/downloads.html

: Proof is often difficult to teach. In this paper, Tall suggests that different forms of proof are appropriate in different contexts, dependent on the particular forms of representation available to the individual, and that these forms become available at different stages of cognitive development. For a young child, proof may be a physical demonstration, long before sophisticated use of the verbal proofs of Euclidean geometry can be introduced successfully to a subset of the school population. Later still, formal proof from axioms involves even greater difficulties that make it appropriate for a few, but impenetrable to many. At this formal stage of development, Tall identifies two different strategies that students adopt to come to terms with formal definition and deduction. Either strategy may be successful, but both are cognitively demanding and prove difficult for many to achieve. This leads to the observation that formal proof is appropriate only for some, that some forms of proof may be appropriate for more, and that, if one allows the simpler representations of proof such as those using physical demonstrations, perhaps some forms of proof are appropriate for (almost) all.

18. Weber, Keith, Students' Difficulties with Proof. MAA Online: Research Sampler. No 8, June 2003. http://www.maa.org/programs/faculty-and-departments/curriculum-department-guidelines-recommendations/teaching-and-learning/research-sampler-8-students-difficulties-with-proof

    Comments: Weber discusses what is meant by the word "proof," in various contexts, and the role that proof plays in mathematics. With this backdrop, he discusses the difficulties that many students experience with learning to prove theorems. Finally he makes some suggestions about how to effectively teach students the concept of proof. The paper is rich in additional references from the literature.